Accuracy of Lattice Boltzmann Method for Aeroacoustic simulations

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The lattice Boltzmann method is used in fluid mechanics since the end of the 90's. Recently some papers have been published about LBM used in flow acoustics. Because the LBM scheme is a weakly compressible one, we can access aerodynamics and acoustics information using one simulation. The purpose of this work is to study the behavior of this information through the LBM and the discrete velocity Boltzmann equation (DVBE). A von Neumann analysis leads us to a modal decomposition of the scheme providing the dispersion and dissipation relation for the shear and propagation modes. We show that in the limit of small Knudsen number, the DVBE dispersion relation and dissipative coefficients perfectly match the theoretical expressions found by Chapman-Enskog analysis. On the other hand, time and space discretization imposed by LBM, introduce a variation of the dispersion relation depending on the wavenumber. The above results are perfectly matched to numerical computations obtained with a 3D code based on the D3Q19 velocity model. An analysis is made on the MRT model with a compressible distribution function. We show that this model does not improve the dispersion relation but can modify the dissipation to improve stability.

I. Introduction

The Lattice Boltzmann method is used in fluid mechanics simulation since the end of the 90s.[1] In order to access the capabilities of this method for aeroacoustic purposes, we have developed a 3D LBM code called L-BEAM (Lattice Boltzmann Equation for Aeroacoustic Modeling), based on the D3Q19 velocity model. Because the Lattice Boltzmann scheme is a weakly compressible one, we can access all the information using one simulation[12].[7] What we pointed out here, is the propagation part of the scheme, that is to say the behavior of a simulated sound wave in terms of energy and phase. After a brief introduction to the Lattice Boltzmann model construction, we first study the accuracy of the scheme on a simple test case to quantify the influence of space and time discretization. Then we focus on the theoretical aspects of the Lattice Boltzmann scheme in terms of dispersion and dissipation. By this study, we want to quantify the acoustic accuracy of the different models known in the literature.

II. Global accuracy of LBM

II.A. Lattice Boltzmann Model

It can be shown[1] that the Boltzmann equation with the BGK collision operator can recover the Navier-Stokes equations. From this equation and by taking a finite number of discrete velocities, we can obtain the discrete velocity Boltzmann equation:

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$$\frac{\partial f_\alpha}{\partial t} + c_{\alpha,i}\frac{\partial f_\alpha}{\partial x_i} = -\frac{1}{\tau}(f_\alpha - f_\alpha^{eq}) \tag{1}$$

The so-called lattice Boltzmann equation can be obtain from Eq.(1) with a space and time discretization and a variable change given by:[5]

$$g_\alpha(\mathbf{x},t) = f_\alpha(\mathbf{x},t) + \frac{\Delta t}{2\tau}(f_\alpha(\mathbf{x},t) - f_\alpha^{eq}(\mathbf{x},t)) \tag{2}$$

then we obtain the Lattice Boltzmann equation on g_α:

$$g_\alpha(\mathbf{x} + c_\alpha \Delta t, t + \Delta t) = g_\alpha(\mathbf{x},t) - \frac{\Delta t}{\tau_g}(g_\alpha(\mathbf{x},t) - g_\alpha^{eq}(\mathbf{x},t) + O(\Delta t^3) \tag{3}$$

with $\tau_g = \tau + \frac{1}{2}$ and g_α^{eq}, the equilibrium distribution function of the form:

$$f_\alpha^{eq}(\mathbf{x},t) = g_\alpha^{eq}(\mathbf{x},t) = \rho\omega_\alpha(1 + \frac{\mathbf{u}.\mathbf{c}_\alpha}{c^2} + \frac{(\mathbf{u}.\mathbf{c}_\alpha)^2}{2c^4} - \frac{|\mathbf{u}|^2}{2c^2}) \tag{4}$$

where $c = \frac{1}{\sqrt{3}}$ is the adimensional sound speed for the D3Q19 velocity model and the macroscopic quantities ρ and $\mathbf{u}$ can be expressed in the form:

$$\rho = \sum_\alpha f_\alpha = \sum_\alpha g_\alpha \tag{5}$$

$$\rho\mathbf{u} = \sum_\alpha \mathbf{c}_\alpha f_\alpha = \sum_\alpha \mathbf{c}_\alpha g_\alpha \tag{6}$$

What we can pointed out here is the error due to space and time discretization in Eq.(3). This error can be easily observed on a simple numerical test case.

II.B. Accuracy of the LBM scheme

To study the accuracy of the scheme, we simulate a 3D gaussian pulse in a periodic domain which propagatesin the air. The analytical solution of this problem is well known[9] and can be written as:

$$\rho'(x,y,z,t) = \frac{\varepsilon}{2\alpha\sqrt{\pi\alpha}}\int_0^\infty e^{-\frac{\xi^2}{4\alpha}} cos(c_0 t\xi) J_0(\xi\eta)\xi^2 d\xi \tag{7}$$

with $\eta = \sqrt{(x-u_0 t)^2 + y^2 + z^2}$ and ε, b_p the initial amplitude of the pulse and the adimensional wavelength respectively. Then we can compute the numerical error between analytical and numerical solution by evaluating the L_2 norm for different resolutions:

$$L_2 = \frac{1}{N}\sum_{i=1}^{N}(p_i^{th} - p_i^{num})^2 \tag{8}$$

Fig(1(b)) show the evolution of the L_2 norm for two kind of real number coding. The first curve is obtained with simple precision and the other one with the commercial code PowerFLOW based on the same velocity model but coded in simple precision. We can see that in the case of simple precision the slope of the L_2 norm decreases for high resolution. For high resolution, a higher number of timesteps is used to keep the same physical time, in this case, the round-off error due to simple precision is accumulating. For the double precision case, the slope of 2.15 is in good agreement with the theoretical second order convergence rate of the LBM scheme.[1]

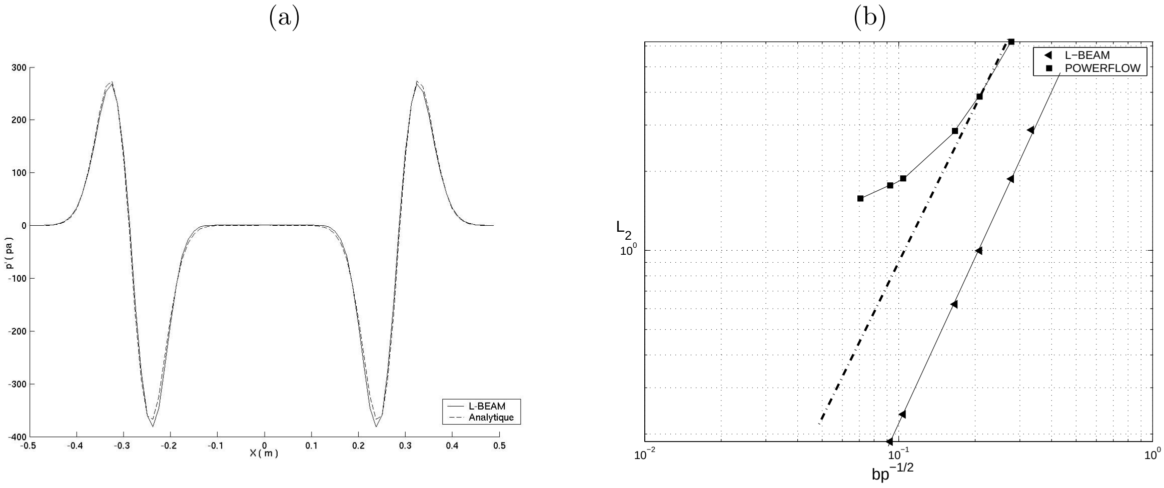

Figure 1. (a) Numerical and analytical solutions of the gaussian pulse for 10 points per wavelength (b) Evolution of the L_2 norm with the resolution.

III. Theoretical behavior

Now, the idea is to study the Boltzmann scheme theoretically to understand the behavior of sound waves. The first thing to do, is to look for plane wave solutions of the scheme. We will proceed to a von Neumann analysis to achieve such a result. This approach has been applied to Lattice Boltzmann scheme first by L.S Luo.[10] His approach was based on successive approximation in k. The idea is here to proceed differently by doing a direct numerical computation of the solutions using a linear algebra library.

III.A. Von Neumann analysis

The von Neumann analysis consists in looking for plane waves solutions of the linearized equations. Here we look for solutions of the form:

$$f'_\alpha = A_\alpha exp(\mathbf{k}.\mathbf{x} - \omega t) \tag{9}$$

To linearize the equation, we first have to consider the distribution functions as a mean part f^0_α and a fluctuating part f'_α, and then to linearize the nonlinear terms. These terms are contained in the equilibrium distribution function (Eq.4). By using a Taylor expansion of this function, we can write:

$$f^{eq}(f^{(0)}_\alpha + f'_\alpha) = f^{eq,(0)}_\alpha + \frac{\partial f^{eq}_\alpha}{\partial f_\beta}\Big|_{f_\beta = f^{(0)}_\beta} f'_\alpha + o({f'_\alpha}^2) \tag{10}$$

Thanks to Eq.(10) we can express the governing equations (1) and (3) in terms of fluctuating part f'_α. Let's see now the behavior of the solution for the different equation.

III.B. The Discrete Velocity Boltzmann Equation

By injecting (9) and (10) in Eq.(1), the DVBE can be rewritten as:

$$i\omega \mathbf{f'} = M^{\mathrm{DVBE}} \mathbf{f'} \tag{11}$$

with $\mathbf{f'}$ the vector of the distribution functions $(f'_1, ..., f'_N)$ and M^{DVBE} a matrix defined by:

$$M^{\mathrm{DVBE}}_{\alpha\beta} = \frac{1}{\tau}\left[\delta_{\alpha\beta} - \frac{\partial f^{eq}_\alpha}{\partial f_\beta}\Big|_{f_\beta = f^{(0)}_\beta}\right] + i\mathbf{k}.\mathbf{c}_\alpha \delta_{\alpha\beta} \tag{12}$$

Eq.(11) is nothing else that an eigenvalue problem in which $i\omega$ are the eigenvalues of matrix M^{DVBE}. The coefficients of matrix M^{DVBE} depend on three parameters which are, the relaxation time τ, the direction of propagation holds by vector $\mathbf{k}[k_x, k_y, k_z]$ and the mean flow $\mathbf{U_0}[U_{0_x}, U_{0_y}, U_{0_z}]$. The evaluation and simplification of the matrix coefficients can be done with Maple, particularly for the second term of Eq.(10), the explicit coefficients are given in Appendix. The solutions of the problem given by Eq.(11) can be calculated numerically with Matlab. This equation tells us that the evolution of the eigenvalues λ_α of M^{DVBE} are equivalent to the evolution of the pulsation $\omega(\mathbf{k})$. Moreover, by proceeding to a Chapman-Enskog expansion of Eq.(11), it can be shown,[11] that the eigenvalues λ_α of matrix M^{DVBE} can be linked to the transport coefficients as follow:

$$Im[\lambda_\alpha^{\pm}(\mathbf{k})] = -Re[\omega^{\pm}(\mathbf{k})] = \mp k[c(\mathbf{k}) \pm U_0(\mathbf{k})] \tag{13}$$

$$Re[\lambda_\alpha^{\pm}(\mathbf{k})] = Im[\omega^{\pm}(\mathbf{k})] = -k^2[\frac{1}{2}\nu(\mathbf{k}) + \xi(k)] \tag{14}$$

$$Im[\lambda_\alpha^{T}(\mathbf{k})] = -Re[\omega^{T}(\mathbf{k})] = -kU_0(\mathbf{k}) \tag{15}$$

$$Re[\lambda_\alpha^{T}(\mathbf{k})] = Im[\omega^{T}(\mathbf{k})] = -k^2\nu(\mathbf{k}) \tag{16}$$

where $\xi = \frac{1}{2}\nu$ in the D3Q19 model, $^{\pm}$ denotes acoustic modes and ^{T} is related to shear modes. Thus, by this result, we get the dispersion and dissipation relation with $Re(i\omega) = Re(\lambda_\alpha)$ and $Im(i\omega) = Im(\lambda_\alpha)$. Fig.(2) shows the evolution of the different modes for $\mathbf{k} = [k_x, 0, 0]$,$\tau = 0.0025$ and $\mathbf{U_0} = [0.12, 0, 0]$.

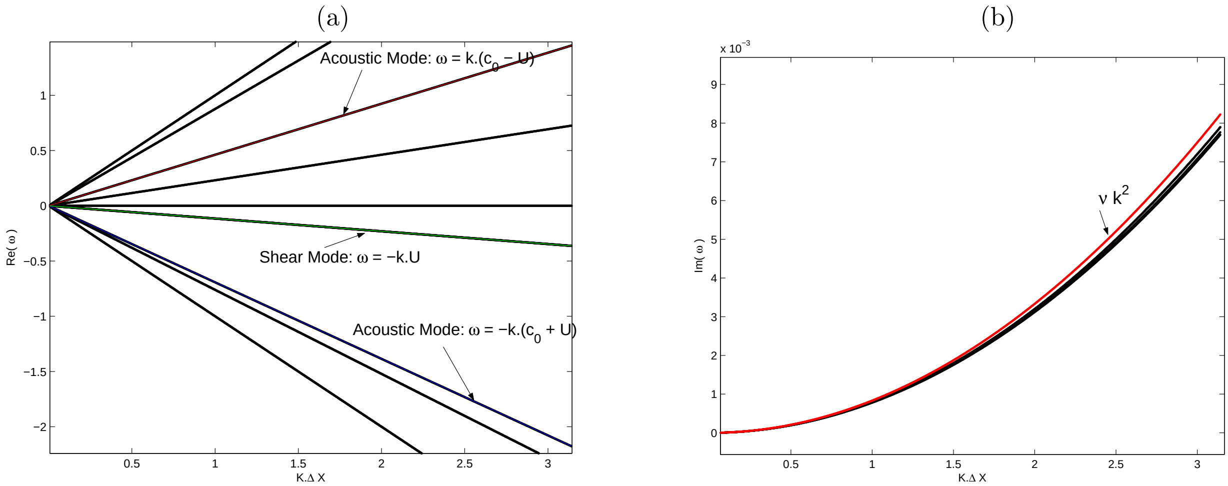

Figure 2. Real and Imaginary part evolution of M^{DVBE} eigenvalues compared to the theoretical curves

Only three modes out of 19 seem to be physical ones. On the real part (dispersion relation), we can recognize the two acoustics modes which match the theoretical curve $\omega = \mathbf{k}(U_0 + c_0)$ and the shear mode $\omega = \mathbf{k}U_0$. These modes are found again on the imaginary part (Dissipation relation) and match the curve of the dissipation due to the air viscosity. Considering these results, the DVBE seems to be exact for acoustic dispersion and dissipation. In fact, by studying the influence of the parameter τ, a larger error occurs for high values of the relaxation time (i.e high values of the Knundsen number). The main idea to point out here, is that in the limit of small Knundsen number, the velocity discretization introduce no error in the behavior of sound waves. We now have to considerate the Lattice Boltzmann Equation to study the influence of space and time discretization.

III.C. The Lattice Boltzmann Equation

We consider here, the Lattice Boltzmann Equation, which contains space and time discretization. The propagation behavior of this scheme has been studied for a linear equilibrium distribution function. We can show that for the particular case of linear LBM with $\tau = 0$, the LBM is equivalent to TLM (Transmission Line Matrix) and that the D2Q4 model predicts the same dispersion results than TLM.[8]
For the nonlinear LBM, the von Neumann analysis of Eq.(3) with $\Delta t = 1$ leads to the expression:

$$e^{-i\omega}\mathbf{g'} = M^{\mathrm{LBM}}\mathbf{g'} \tag{17}$$

with the new matrix $M^{\mathrm{LBM}} = A^{-1}[I - \frac{1}{\tau_g}N^{\mathrm{LBM}}]$ where I is the identity matrix and A and N^{LBM} defined by:

$$A_{\alpha\beta} = e^{i\mathbf{k}.\mathbf{c}_\alpha}\delta_{\alpha\beta} \tag{18}$$

$$N^{\mathrm{LBM}}_{\alpha\beta} = \delta_{\alpha\beta} - \frac{\partial g^{eq}_\alpha}{\partial g_\beta}\Big|_{g_\beta = g^{(0)}_\beta} \tag{19}$$

We obtain a new eigenvalue problem containing exponential terms. The only difference between Eq.(11) and Eq.(17) is the space and time discretization. Now the adimensional wavenumber vector $\mathbf{k}$ represents the number of points per wavelength by:

$$\tilde{\mathbf{k}} = \mathbf{k}\Delta x = \frac{2\pi}{N_{ppw}} \tag{20}$$

In the following, the limit of $\tilde{k} = \pi$ will corresponds to 2 points per wavelength. The relation between eigenvalues and transport coefficients (Eqs.(13,14,15,16) still the same, replacing λ_α by $\ln\lambda_\alpha$.

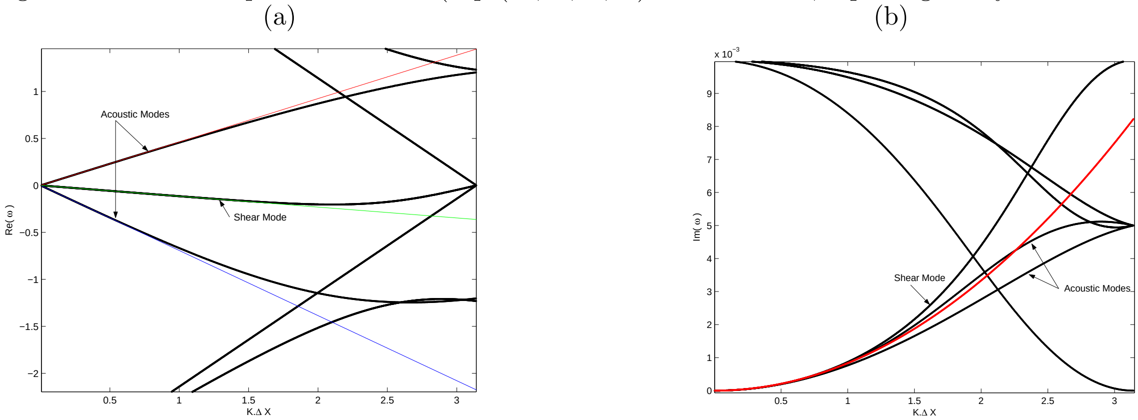

Figure 3. Real and imaginary part evolution of the eigenvalues for LBM.

Considering the same case than in III.B, we can plot (Fig.3) the evolution of the transport coefficients with $\mathbf{k}$ and compare it to their theoretical ones. It appears clearly, that the space and time discretization introduce an error in the evaluation of dispersion and dissipation. We can recognize the same physical modes than in III.B but with a different behavior. The sound speed is underestimated which induces a delay in the wave propagation, and the viscosity is overestimated for the shear modes but underestimated for the shear modes. These results are recovered with the numerical computation. We have simulated a plane wave in a periodical domain for different resolution with the D3Q19 model. Fig.4 compares the evolution of the sound speed for theoretical and numerical results done with L-BEAM and PowerFLOW. The numerical results match perfectly with the above prediction.

III.D. The Multiple Relaxation Time model

The multiple relaxation time model,[36] has been presented recently and is an alternative to the standard BGK model. The idea is not to present this model in details but to resume the main features for our purposes. The MRT model presents a different relaxation time for each momentum. We have to define as much momentum as discrete velocities, which introduce a correspondence matrix P transforming distribution function vector into momentum ones. By this way, the collision step of the algorithm must be done in the momentum space, whereas the propagation step is done in the physical space. The relaxation time in the collision term of Eq.(1) is then replaced by a diagonal matrix S containing the different relaxation times. This model allows us to control independently the relaxation of the different moments. The equation of such a model can be written as:

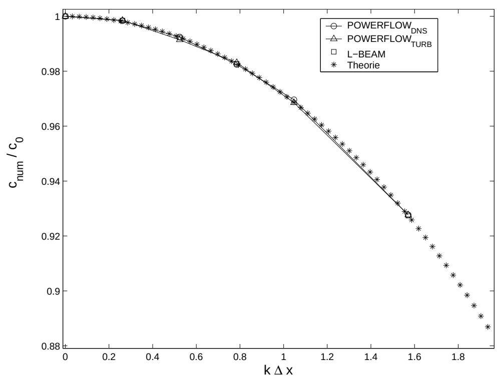

Figure 4. Evolution of sound speed with the resolution for $U_0 = 0$. c_{num} corresponds to the simulated wave speed and the theoretical speed $c(k)$. c_0 is the speed of sound in the air.

$$\mathbf{g}(\mathbf{x}+\mathbf{c},t+1) = \mathbf{g}(\mathbf{x},t) - P^{-1}S[\mathbf{m}(\mathbf{x},t) - \mathbf{m_{eq}}(\mathbf{x},t)] \tag{21}$$

where $\mathbf{m}$ is the momentum vector such as $\mathbf{m} = P\mathbf{f}$ and S defined as follow:

$$S = diag[\frac{1}{\tau_1}, ..., \frac{1}{\tau_N}] \tag{22}$$

In classical MRT model the equilibrium distribution function is quite different than its Eq.(4) form. It could be written as:

$$f_\alpha^{eq}(\mathbf{x},t) = \omega_\alpha \Big[\rho + \rho_0(\frac{\mathbf{u}.\mathbf{c}_\alpha}{c^2} + \frac{(\mathbf{u}.\mathbf{c}_\alpha)^2}{2c^4} - \frac{|\mathbf{u}|^2}{2c^2})\Big] \tag{23}$$

which is similar to an incompressible model.[2] From this, the classical MRT model evaluate vector $\mathbf{m}^{eq}$ with:

$$\mathbf{m}^{eq} = P\mathbf{f}^{eq} \tag{24}$$

This kind of incompressible model has been studied in the literature.[4] In this work, we will study the acoustic behavior of the MRT model with an equilibrium distribution function defined by Eq.(4). Applying relation (10) to $\mathbf{m}$ the von Neumann analysis of Eq.(21) leads to the new eigenvalue problem:

$$e^{-i\omega}\mathbf{g'} = M^{\text{MRT}}\mathbf{g'} \tag{25}$$

with the new matrix $M^{\text{MRT}} = A^{-1}[I - P^{-1}SN^{\text{MRT}}P]$ where N^{MRT} defined by:

$$N_{\alpha\beta}^{\text{MRT}} = \delta_{\alpha\beta} - \frac{\partial m_\alpha^{eq}}{\partial m_\beta}\Big|_{m_\alpha = m_0} \tag{26}$$

Because the MRT model must recover the BGK model for $S = \frac{1}{\tau_g}I$, the equality of Eq.(17) and Eq(25) leads to:

$$N^{\text{MRT}} = PN^{\text{LBM}}P^{-1} \tag{27}$$

According to this result, we can solve the eigenvalue problem (25), just by knowing S and P. The coefficients of P and S are given in the literature.[6] The relaxation times are chosen in order to control separately the bulk and shear viscosity. The evolution of the MRT modes are presented on Fig(5).

It is important to note that the MRT model gives the same evolution for dispersion relation than the BGK model. The differences are in the dissipation relation, because we can control momentum relaxation independently, we can control acoustic dissipation and shear dissipation independently. We will see in the following that it can be helpful for stability problem.

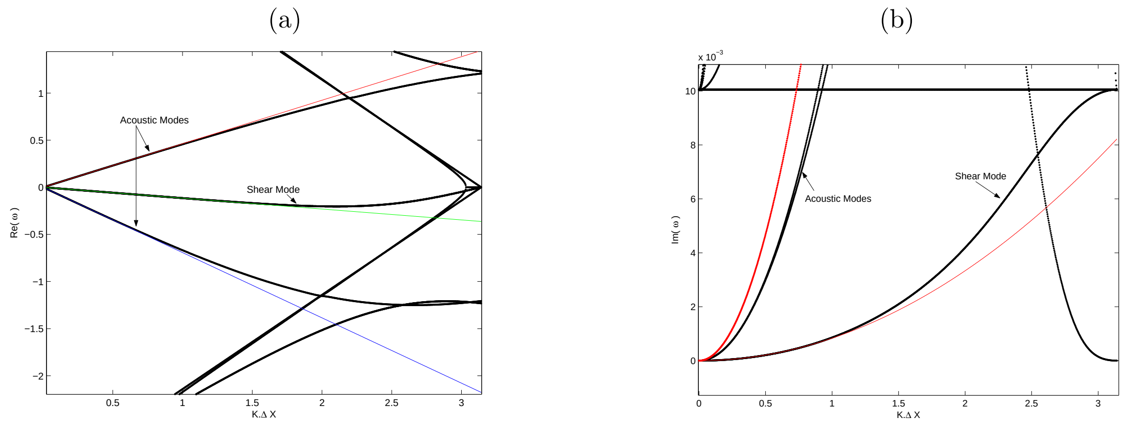

Figure 5. Real and Imaginary part evolution of the eigenvalues for the MRT model.

III.E. Stability

Until now, we have considered a propagation along the x direction ($\mathbf{k} = [k_x, 0, 0]$). Let's see now, the influence of the propagation direction. We introduce new parameters which are the angles ϕ and θ of the spherical coordinates as illustrated on Fig(6).

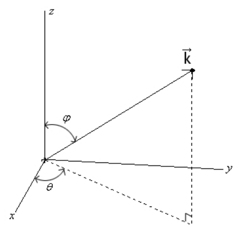

Figure 6. Spherical coordinates for vector k.

By studying the influence of ϕ, θ and U_0 on the eigenvalues of the different matrix M, we notice that in certain cases, these eigenvalues become negatives (Fig.7), denoting a non-isotropy of the scheme.

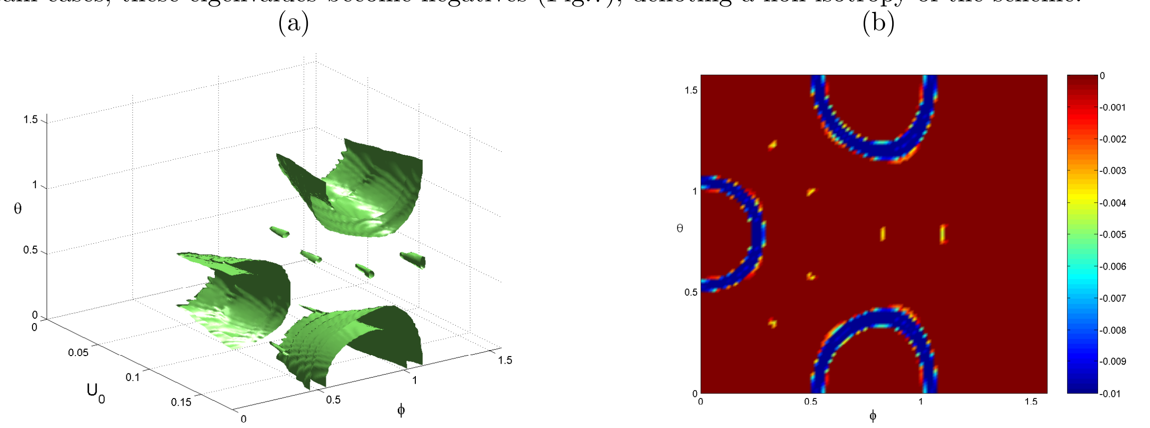

Figure 7. Repartition of $min(\ln \lambda_\alpha)$.(a) Isosurfaces of negatives eigenvalues. (b) Visualization for the plane $V = 0.16$

In terms of stability, this means that the simulation becomes unstable. Let's now focus on a particular case to see the evolution of the modes in such a situation. Considering an unstable configuration taken from Fig.7, the dispersion and dissipation analysis allows us to highlight some particular behavior. Fig.8 shows that the eigenvalues become negative on a small interval of high k value. On the dispersion curve (real part), we see that the acoustic and shear modes become very close. We can interpret this as an energy transfer between the two modes involving a negative dissipation (i.e energy gain). But to make the energy transfer possible, the modes could be able to see each other, which is not possible with linear phenomenon. Actually, this could be explained by a non-normality condition of the discrete velocity basis, induced by the fact that M is a full matrix.

(a) (b)

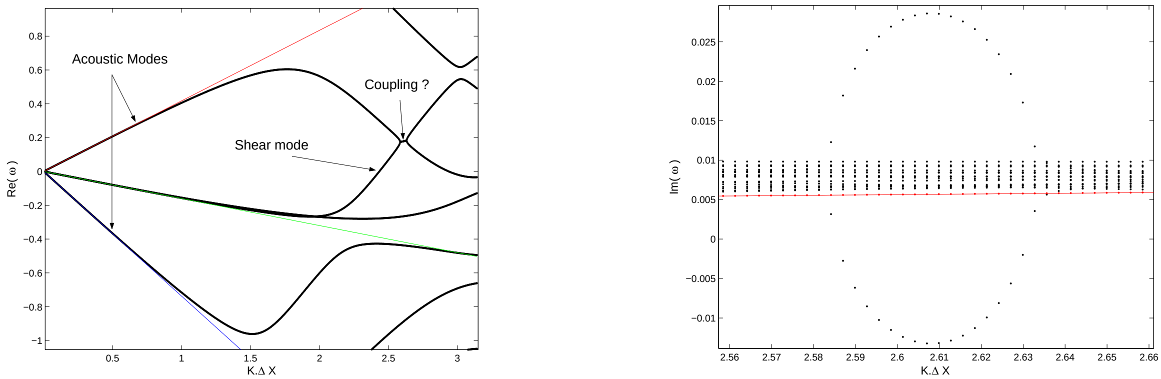

Figure 8. Real and Imaginary part evolution of the eigenvalues for LBM with $U_0 = 0.16$, $\theta = 20.6$ and $\phi = 45.8$.

(a) (b)

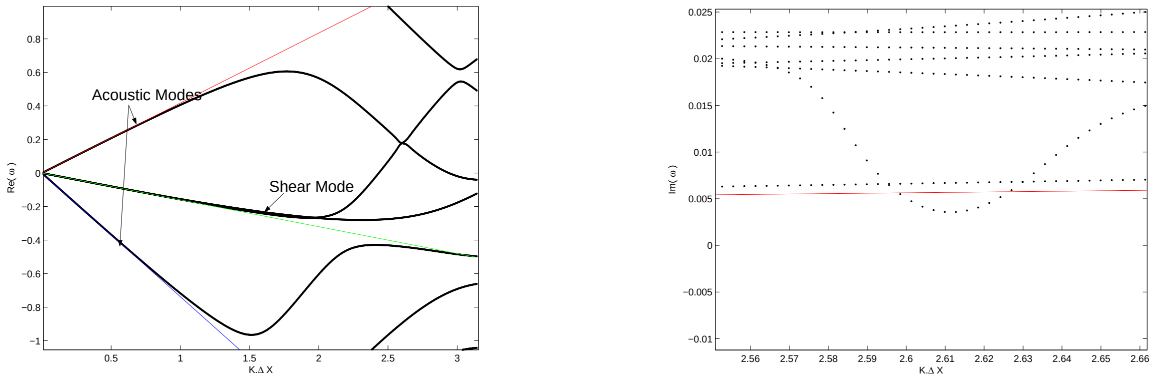

Figure 9. Real and Imaginary part evolution of the eigenvalues for MRT with $U_0 = 0.16$, $\theta = 20.6$ and $\phi = 45.8$.

If we compare the behavior of the different model (BGK Fig.(8), MRT Fig.(9)), we notice that the MRT model is more stable than the BGK one. This is due to the overdissipation of the acoustic modes in the MRT model. Indeed, the acoustic mode dissipated by the bulk viscosity, is less energetic when it is coupled with the shear mode, so that the eigenvalues still positives. So, it is important to show that, if the MRT model allows us to control independently acoustic modes and shear mode, the stability problem leads us to control the acoustic modes in a dissipative way which is not suitable for aeroacoustic computations.

IV. Conclusion

In this paper, we have studied the LBM scheme for acoustic purposes. By studying the accuracy of the scheme, we have pointed out the global error due to space and time discretization and seen its effects on acoustic propagation. We have shown that the von Neumann analysis made on a compressible scheme could give us information about dispersion and dissipation. By studying the MRT model, we have seen that the bulk viscosity could be controlled independently but that its effects on acoustic waves were not convenient for acoustic purposes. The stability study has shown that the different modes could interfere, involving unstable situations. It appears clearly that the LBM models have to be a compromise between stability and dissipation. The fact that both problems have something to do with viscosity, leads us to think to an other way to increase stability. These considerations will be parts of our future work.

Acknowledgments

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References

[1] S. Chen, G. Doolen, *Lattice Boltzmann method for fluid flows*, Annu. Rev. Fluid Mech. 161 (1998)329.

[2] Xiaoyi He, Li-Shi Luo, *Lattice Boltzmann Model for the Incompressible Navier-Stokes Equation*, J. Stat. Phys. 88. pp927-944, 1997.

[3] P. Lallemand and Li-Shi-Luo, *Theory of the lattice Boltzmann method: Acoustic and thermal properties in two and three dimension*, Ph. Review E, 68, 036706, 2003

[4] P. Lallemand and Li-Shi-Luo, *Theory of the lattice Boltzmann method: Dispersion, dissipation, isotropy, Galilean invariance and stability*, Ph. Review E, 61, 06, 2000

[5] P.J. Dellar, *Bulk and shear viscosities in lattice Boltzmann equations*, Phys. Rev.E, 64,031203.

[6] D.d'Humière, I. Ginzburg, M. Krafczyk, P. Lallemand, L.S Luo, *Multiple relaxation time lattice Boltzmann models in three dimensions*, Phil. Trons. R. Soc. Lon.A, 360 pp 437-451, 2002

[7] A. Wilde, *Application of the Lattice-Boltzmann method in flow acoustics*, 4th SWING Aeroacoustic Workshop, Aachen, Feb. 26, 2004

[8] P.O Lüthi, *Lattice wave automata*, PhD. Thesis, Université de Genève, 1998

[9] C.K.W. Tam and J.C. Webb (1993) *Dispersion relation preserving finite difference schemes for computational acoustics.*, J. Comput. Phys., vol. 107, pp 262-281.

[10] L.S Luo, *Lattice-gas Automata and Lattice Boltzmann equations for two-dimensional hydrodynamics*, PhD Thesis , Georgia Institute of Technology, April, 1993.

[11] L.D. Landau, E.M. Lifshitz *Fluid Mechanics*, 2nd edition, Pergamon Oxford, 1987.

[12] D. Ricot, V. Maillard, C. Bailly, *Numerical simulation of unsteady cavity flow using Lattice Boltzmann Method*, 8th AIAA/CEAS Aeroacoustics conference, Breckenridge, 2002 AIAA paper 2002-2532.